DP Analysis & Approaches HL

Maths with Lemon

Lesson 9.6HL – Integration of Trigonometric Functions

Mathematician: _____

HL 5.15 – Syllabus and Key Ideas

Content	Guidance
Indefinite integrals of derivatives of $\tan x, \sec x, \csc x, \cot x$, inverse trig functions and exponential functions.	Use reverse differentiation, linear substitution, completing the square and partial fractions.

Useful Formulae

$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \csc^2 x\,dx = -\cot x + C,$$

$$\int \sec x \tan x\,dx = \sec x + C, \qquad \int \csc x \cot x\,dx = -\csc x + C,$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \qquad \int \frac{dx}{1+x^2} = \arctan x + C,$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1).$$

Warm Up

Find the gradient of $y = x + 2\cot x$ at $x = \pi/4$.

$$y' = 1 - 2\csc^2 x, \quad y'(\pi/4) = 1 - 2(2) = \boxed{-3}.$$

Using Old Methods or Formula Booklet

$$\int \frac{1 + 2\sin x}{\cos^2 x}\,dx = \int (\sec^2 x + 2\sec x \tan x)\,dx = \boxed{\tan x + 2\sec x + C}.$$

Extra Practice

Worked Example	You Try
$\int \frac{2 + \sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$ Split the fraction: $2/\sqrt{1-x^2} + 1$. $\boxed{2\arcsin x + x + C}$	$\int \frac{2\sqrt{1-x^2} - 3}{\sqrt{1-x^2}}\,dx$ *Show your working.* . . . .

Worked Example	You Try
$$\int \frac{2+3\sin x}{\cos^2 x}\,dx$$	$$\int \frac{3-\cos x}{\sin^2 x}\,dx$$
$2\sec^2 x + 3\sec x\tan x$. $\boxed{2\tan x + 3\sec x + C}$	*Show your working.* . . . .

Worked Example	You Try
$$\int \sec x(2\sec x + 3\tan x)\,dx$$	$$\int \csc x(5\csc x + 2\cot x)\,dx$$
$2\sec^2 x + 3\sec x\tan x$. $\boxed{2\tan x + 3\sec x + C}$	*Show your working.* . . . .

Worked Example	You Try
$$\int \frac{3}{\sqrt{1-x^2}}\,dx$$	$$\int \frac{4}{1+x^2}\,dx$$
Use inverse sine formula. $\boxed{3\arcsin x + C}$	*Show your working.* . . . .

Worked Example	You Try
$$\int 2^x\,dx$$	$$\int 3^x\,dx$$
Apply exponential integral formula. $\boxed{\frac{2^x}{\ln 2} + C}$	*Show your working.* . . . .

Composition of Linear Functions

For a linear inner function $u = ax + b$, remember $du = a\,dx$:

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C.$$

Guided example:

$$\int 6\sec^2(3x+1)\,dx = 2\tan(3x+1) + C.$$

Worked Example	You Try
$\int 8\csc^2(2x-1)\,dx$ Let $u = 2x-1$, $du = 2dx$. $\boxed{-4\cot(2x-1) + C}$	$\int 9\sec^2(3x+1)\,dx$ *Show your working.* . . . .

Worked Example	You Try
$\int 3^{2x}\,dx$ $u = 2x$. $\boxed{\dfrac{3^{2x}}{2\ln 3} + C}$	$\int 2^{5x}\,dx$ *Show your working.* . . . .

Worked Example	You Try
$\int \dfrac{3}{\sqrt{1-9x^2}}\,dx$ Let $u = 3x$. $\boxed{\arcsin(3x) + C}$	$\int \dfrac{10}{\sqrt{1-25x^2}}\,dx$ *Show your working.* . . . .

Worked Example	You Try
$$\int 5\csc^2(x/3)\,dx$$ $u = x/3$. $\boxed{-15\cot(x/3) + C}$	$$\int 3\sec^2(x/4)\,dx$$ *Show your working.* . . . .

Worked Example	You Try
$$\int \frac{1}{1+4x^2}\,dx$$ Let $u = 2x$. $\boxed{\frac{1}{2}\arctan(2x) + C}$	$$\int \frac{1}{1+16x^2}\,dx$$ *Show your working.* . . . .

Integrals Resulting in arcsin **and** arctan

Complete the square before applying the formula.

$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \boxed{\frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C}.$$

Worked Example	You Try
$$\int \frac{dx}{x^2+4x+5}$$ $x^2+4x+5 = (x+2)^2+1$. $\boxed{\arctan(x+2) + C}$	$$\int \frac{dx}{x^2-6x+10}$$ *Show your working.* . . . .

Worked Example	You Try
$$\int \frac{dx}{x^2+2x+10}$$ $(x+1)^2+9$. $\boxed{\frac{1}{3}\arctan(\frac{x+1}{3})+C}$	$$\int \frac{dx}{x^2+4x+20}$$ *Show your working.* . . . . .

Worked Example	You Try
$$\int \frac{dx}{x^2+6x+11}$$ $(x+3)^2+2$. $\boxed{\frac{1}{\sqrt{2}}\arctan(\frac{x+3}{\sqrt{2}})+C}$	$$\int \frac{dx}{x^2-10x+30}$$ *Show your working.* . . . . .

Worked Example	You Try
$$\int \frac{dx}{\sqrt{12-4x-x^2}}$$ $12-4x-x^2=16-(x+2)^2$. $\boxed{\arcsin(\frac{x+2}{4})+C}$	$$\int \frac{dx}{\sqrt{8+2x-x^2}}$$ *Show your working.* . . . . .

Worked Example	You Try
$$\int \frac{dx}{\sqrt{1+4x-x^2}}$$ $1+4x-x^2=5-(x-2)^2$. $\boxed{\arcsin(\frac{x-2}{\sqrt{5}})+C}$	$$\int \frac{dx}{\sqrt{2-2x-x^2}}$$ *Show your working.* . . . . .

Partial Fraction Decomposition

Guided example:

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3}.$$

Thus

$$\boxed{\int \frac{dx}{x^2-5x+6} = \ln\left|\frac{x-3}{x-2}\right| + C}.$$

Worked Example	You Try
$\int \frac{2x+4}{x^2+4x+3}\,dx$ Denominator derivative is $2x+4$. $\boxed{\ln\lvert x^2+4x+3\rvert + C}$	$\int \frac{x+1}{x^2+x-6}\,dx$ *Show your working.* . . . .

Worked Example	You Try
$\int \frac{4}{x^2-2x-3}\,dx$ $\frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1}$. $\boxed{\ln\lvert\frac{x-3}{x+1}\rvert + C}$	$\int \frac{1}{x^2+5x+6}\,dx$ *Show your working.* . . . .

Worked Example	You Try
$\int \frac{3}{2x^2+x-1}\,dx$ $2x^2+x-1 = (2x-1)(x+1)$. $\frac{3}{(2x-1)(x+1)} = \frac{2}{2x-1} - \frac{1}{x+1}$. $\boxed{\ln\lvert\frac{2x-1}{x+1}\rvert + C}$	$\int \frac{2}{3x^2+4x+1}\,dx$ *Show your working.* . . . .

Detailed Markscheme – You Try

Extra Practice

1. $\displaystyle\int \frac{2\sqrt{1-x^2}-3}{\sqrt{1-x^2}}\,dx$

$$\boxed{2x - 3\arcsin x + C}$$

2. $\displaystyle\int \frac{3-\cos x}{\sin^2 x}\,dx$

$$\boxed{-3\cot x + \csc x + C}$$

3. $\displaystyle\int \csc x(5\csc x + 2\cot x)\,dx$

$$\boxed{-5\cot x - 2\csc x + C}$$

4. $\displaystyle\int \frac{4}{1+x^2}\,dx$

$$\boxed{4\arctan x + C}$$

5. $\displaystyle\int 3^x\,dx$

$$\boxed{\frac{3^x}{\ln 3} + C}$$

Composition of Linear Functions

1. $\displaystyle\int 9\sec^2(3x+1)\,dx$

$$\boxed{3\tan(3x+1) + C}$$

2. $\displaystyle\int 2^{5x}\,dx$

$$\boxed{\frac{2^{5x}}{5\ln 2} + C}$$

3. $\displaystyle\int \frac{10}{\sqrt{1-25x^2}}\,dx$

$$\boxed{2\arcsin(5x) + C}$$

4. $\displaystyle\int 3\sec^2(x/4)\,dx$

$$\boxed{12\tan(x/4) + C}$$

5. $\displaystyle\int \frac{1}{1+16x^2}\,dx$

$$\boxed{\frac{1}{4}\arctan(4x) + C}$$

Inverse Trigonometric Integrals

1. $\displaystyle\int \frac{dx}{x^2-6x+10}$

$$\boxed{\arctan(x-3)+C}$$

2. $\displaystyle\int \frac{dx}{x^2+4x+20}$

$$\boxed{\frac{1}{4}\arctan(\frac{x+2}{4})+C}$$

3. $\displaystyle\int \frac{dx}{x^2-10x+30}$

$$\boxed{\frac{1}{\sqrt{5}}\arctan(\frac{x-5}{\sqrt{5}})+C}$$

4. $\displaystyle\int \frac{dx}{\sqrt{8+2x-x^2}}$

$$\boxed{\arcsin(\frac{x-1}{3})+C}$$

5. $\displaystyle\int \frac{dx}{\sqrt{2-2x-x^2}}$

$$\boxed{\arcsin(\frac{x+1}{\sqrt{3}})+C}$$

Partial Fractions

1. $\displaystyle\int \frac{x+1}{x^2+x-6}\,dx$

$$\boxed{\frac{3}{5}\ln|x-2|+\frac{2}{5}\ln|x+3|+C}$$

2. $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

$$\boxed{\ln|\frac{x+2}{x+3}|+C}$$

3. $\displaystyle\int \frac{2}{3x^2+4x+1}\,dx$

$$\boxed{\ln|\frac{3x+1}{x+1}|+C}$$



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Mathematics made clear.