Lesson 10.3: Separable and Homogeneous Differential Equations

Worksheet – Analytical Solutions at the End

Mathematician: _____

Instructions: Show full analytical working. Give general solutions unless an initial condition is specified. Check constant solutions when dividing by a variable. **Part A: Separable differential equations**

1. Solve $\frac{dy}{dx} = xy$ given $y > 0$.

2. Solve $\frac{dy}{dx} = y + 1$.

3. Solve $\frac{dy}{dx} = x^3y^3$ and identify any constant solution.

4. Solve $\frac{dy}{dx} = 2y$ given $y(0) = 4$.

5. Solve $\frac{dy}{dx} = x^2y^2$ for $y > 0$.

6. Solve $\frac{dy}{dx} = \frac{y}{x}$ for $x > 0, y > 0$.

7. Find the solution of $\frac{dy}{dx} = \frac{\sin x}{y}$ with $y(0) = 10$.

Part B: Homogeneous differential equations

8. Solve $\frac{dy}{dx} = \frac{x+y}{x}$ for $x > 0$.

9. Solve $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ for $x > 0, y \neq 0$.

10. Solve $\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}}$ for $x > 0, y > 0$.

11. Solve $\frac{dy}{dx} = 1 + \frac{2y}{x}$ for $x \neq 0$.

12. Solve $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$ for $x > 0, y \neq 0$.

Part C: Applications and challenge

13. A radioactive sample has initial mass 25 g and initial decay rate 5 g/s. Assuming decay rate is proportional to mass, find the model and its half-life.

14. A bacterial population starts at 2000 with an initial growth rate of 500 bacteria per minute. Find its population after 10 minutes.

15. Solve $\frac{dN}{dt} = kN(a - N)$ for $a, k > 0$ and $0 < N(0) = N_0 < a$. Interpret a.

16. Solve $\frac{dy}{dx} = \frac{(2x + y)^2}{4x^2}$ with $y(1) = 0$.

17. Challenge: let $z = 1/y$ in $y' = xy(y-1)$, with $y(0) = 1/3$. Show that $z' = x(z-1)$ and find $y(x)$.

18. Challenge: let $u = y + 1/x$ in $x^2y' = xy - x + 2$, for $x \neq 0$. Show $xu' = u - 1$ and find $y(x)$.

Detailed Analytical Solutions

1. $dy/y = x\,dx$; hence $\ln y = x^2/2 + C$, so $\boxed{y = Ae^{x^2/2},\ A > 0}$.

2. $dy/(y+1) = dx$, so $\ln|y+1| = x + C$, giving $\boxed{y = Ae^x - 1}$. This includes $y = -1$ for $A = 0$.

3. $y^{-3}dy = x^3dx$. Thus $-1/(2y^2) = x^4/4 + C$, or $\boxed{y^{-2} = C - x^4/2}$ for $y \neq 0$. Also $\boxed{y = 0}$.

4. $dy/y = 2dx$ gives $y = Ae^{2x}$. Since $y(0) = 4$, $\boxed{y = 4e^{2x}}$.

5. $y^{-2}dy = x^2dx$, so $-1/y = x^3/3 + C$, giving $\boxed{y = 1/(C - x^3/3)}$ on intervals where the denominator is positive.

6. $dy/y = dx/x$ gives $\ln y = \ln x + C$ and $\boxed{y = Ax,\ A > 0}$.

7. $y\,dy = \sin x\,dx$, so $y^2/2 = -\cos x + C$. With $y(0) = 10$, $50 = -1 + C$, hence $\boxed{y = \sqrt{102 - 2\cos x}}$.

8. $y = vx$ implies $v + xv' = 1 + v$, hence $v' = 1/x$. Thus $v = \ln x + C$ and $\boxed{y = x(\ln x + C)}$.

9. $y = vx$ implies $v + xv' = 1/v - v$, so $xv' = (1 - 2v^2)/v$. Separate $v/(1 - 2v^2)\,dv = dx/x$, integrate $-\frac{1}{4}\ln|1 - 2v^2| = \ln x + C$. Thus $1 - 2v^2 = Cx^{-4}$ and $\boxed{2x^2y^2 = x^4 - C}$. Constant-ratio solutions $v = \pm 1/\sqrt{2}$ are included by $C = 0$.

10. With $y = vx$ and $v > 0$, $v + xv' = v + \sqrt{v}$. Thus $dv/\sqrt{v} = dx/x$, giving $2\sqrt{v} = \ln x + C$. Hence $\boxed{y = \dfrac{x}{4}(\ln x + C)^2}$ on intervals where $\ln x + C > 0$.

11. $y = vx$ gives $v + xv' = 1 + 2v$, so $dv/(1+v) = dx/x$. Thus $1 + v = Cx$ and $\boxed{y = Cx^2 - x}$ (valid on intervals not crossing 0).

12. $y = vx$ gives $v + xv' = v + 1/v$, so $v\,dv = dx/x$. Therefore $v^2 = 2\ln x + C$ and $\boxed{y^2 = x^2(2\ln x + C)}$.

13. $m' = -km$. Since $m'(0) = -5 = -25k$, $k = 1/5$. Therefore $\boxed{m(t) = 25e^{-t/5}}$. Setting $m = 12.5$ gives $e^{-t/5} = 1/2$, so $\boxed{t = 5\ln 2 \approx 3.47\text{ s}}$.

14. $N' = kN$ with $500 = 2000k$ gives $k = 1/4$. Hence $N = 2000e^{t/4}$ and $\boxed{N(10) = 2000e^{2.5} \approx 24365}$ bacteria (about 24000 to the nearest thousand).

15. Separate $dN/[N(a-N)] = kdt$. Partial fractions give $\frac{1}{a}[\ln N - \ln(a-N)] = kt + C$ for $0 < N < a$. Hence $N/(a-N) = Ce^{akt}$ and $\boxed{N(t) = a/[1 + ((a - N_0)/N_0)e^{-akt}]}$. The parameter a is the limiting carrying capacity.

16. Let $v = y/x$ so $v + xv' = (v+2)^2/4$. Then $xv' = v^2/4 + 1$, and $4\,dv/(v^2+4) = dx/x$. Integrating, $2\arctan(v/2) = \ln x + C$. Since $y(1) = 0$, $C = 0$. Thus $\boxed{y = 2x\tan(\frac{1}{2}\ln x)}$ on the interval containing $x = 1$ where the tangent is defined.

17. $z = 1/y$ gives $z' = -y'/y^2 = -x(y-1)/y = x(z-1)$. Therefore $dz/(z-1) = x\,dx$, yielding $z - 1 = Ce^{x^2/2}$. Since $z(0) = 3$, $C = 2$. Thus $\boxed{y = 1/(1 + 2e^{x^2/2})}$.

18. $u = y + 1/x$ gives $u' = y' - 1/x^2$. From $x^2y' = xy - x + 2$ we obtain $x^2u' = xy - x + 1 = xu - x$, so $xu' = u - 1$. Separate $du/(u-1) = dx/x$, yielding $u - 1 = Cx$. Hence $\boxed{y = 1 + Cx - 1/x}$. The case $u = 1$ is included with $C = 0$.



Maths with Lemon

Mathematics made clear.