Lesson 10.2: First-Order Differential Equations

Euler's Method – Theory and Worked Examples

IB AA HL AHL 5.18: First-order differential equations and numerical solutions of $\frac{dy}{dx} = f(x, y)$ by Euler's method, using $x_{n+1} = x_n + h$.

1. What is a differential equation?

A differential equation relates an unknown function to one or more of its derivatives. Its **order** is the highest derivative present. A first-order example is $\frac{dy}{dx} = x + y$; a second-order example is $\frac{d^2y}{dx^2} = x - y$. A differential equation is **linear** if y and its derivatives occur only to the first power and are not multiplied together; thus $y' + 2y = x$ is linear, but $(y')^2 + y = y^2 - 4$ is nonlinear. **2. General and particular solutions**

A **general solution** contains an arbitrary constant C. An initial or boundary condition determines a **particular solution**.

$$\frac{dy}{dx} = e^{2x} \quad \Longrightarrow \quad y = \frac{1}{2}e^{2x} + C.$$

If $y(0) = 3$, then $3 = \frac{1}{2} + C$, so $C = \frac{5}{2}$ and

$$\boxed{y = \frac{1}{2}e^{2x} + \frac{5}{2}}.$$

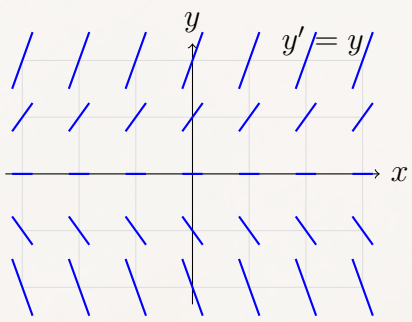
Differentiate the result to check it satisfies the original equation. **3. Translating words into differential equations**

If Q changes at a rate proportional to $g(Q)$, write $\frac{dQ}{dt} = kg(Q)$, where k is a constant. Use a negative sign for decay.

Statement	Model
Bacterial growth proportional to population B	$\frac{dB}{dt} = kB$
Human height grows at a rate inversely proportional to h	$\frac{dh}{dt} = \frac{k}{h}$
Water leaves a bucket at a rate proportional to $\sqrt{V}$	$\frac{dV}{dt} = -k\sqrt{V}$
Infection growth proportional to infected I and uninfected $N - I$	$\frac{dI}{dt} = kI(N - I)$
Rumour spread proportional to $R(N - R)$ and inversely proportional to time	$\frac{dR}{dt} = \frac{kR(N-R)}{t}$, $t > 0$

4. Slope fields (direction fields)

For $y' = f(x, y)$, draw a short segment of gradient $f(x, y)$ at each point (x, y). A solution curve follows these segments and passes through the given initial point. If $y' = y$, slopes depend only on height; if $y' = -xy$, slopes are zero along both axes.



5. Deriving Euler's method

Starting at (x_n, y_n), approximate the curve by its tangent over a horizontal step h. Since gradient is $f(x_n, y_n)$,

$$\Delta y \approx h f(x_n, y_n).$$

Thus

$$\boxed{x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h f(x_n, y_n).}$$

This is the **forward Euler method**. It uses the slope at the *beginning* of each interval.

6. Complete worked example

Estimate $y(1)$ for $y' = x + 2y$, $y(0) = 2$, $h = 0.2$.

$$y_{n+1} = y_n + 0.2(x_n + 2y_n).$$

n	x_n	y_n	$x_n + 2y_n$
0	0.0	2.00000	4.00000
1	0.2	2.80000	5.80000
2	0.4	3.96000	8.32000
3	0.6	5.62400	11.84800
4	0.8	7.99360	16.78720
5	1.0	11.35104	–

For example, $y_1 = 2 + 0.2(4) = 2.8$, $y_2 = 2.8 + 0.2(5.8) = 3.96$. Therefore $\boxed{y(1) \approx 11.35104}$.

7. Euler's method with tabulated velocity

If $\frac{ds}{dt} = v(t)$, Euler's method gives $s_{n+1} = s_n + h v_n$. This is a **left-endpoint rectangle estimate** for displacement. For speeds $0, 6, 12, 19, 24, 27$ m/s at times $0, 3, 6, 9, 12, 15$ seconds, use the first five speeds:

$$s(15) - s(0) \approx 3(0 + 6 + 12 + 19 + 24) = \boxed{183 \text{ m}}.$$

8. Accuracy, limitations and calculator use

Smaller $|h|$ generally improves accuracy but requires more steps. Forward Euler may underestimate or overestimate depending on curvature, and errors accumulate. A GDC spreadsheet can use columns $n, x_n, y_n, f(x_n, y_n)$ with formulas $x_{n+1} = x_n + h$, $y_{n+1} = y_n + h f(x_n, y_n)$. Do not substitute the new x or new y into the current slope.

9. Exam checklist

Identify $f(x, y)$, initial point and h; calculate the number of steps $N = (x_{\text{target}} - x_0)/h$; show the recurrence; keep guard digits; state the approximation at the requested x with suitable rounding and units.



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